LAGRANGIAN DUALITY

Pei-Yuan Wu
Electrical Engineering Department
National Taiwan University

OUT LINE

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- Geometric Interpretation of the Lagrangian Dual
- Weak Duality
- Strong Duality
  - Example

LAGRANGIAN DUALITY

Primal Problem P

Minimize $f(x)$

Subject to:

$g_i(x) \leq 0, i = 1, \dots, m$

$h_i(x) = 0, i = 1, \dots, \ell$

Variables: $x \in \mathcal{X}$

Lagrangian Dual Problem D

Lagrange multipliers

Maximize $\theta(\boldsymbol{u}, \mathbf{v}) = \inf_{x \in \mathcal{X}} \{f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{i=1}^{\ell} v_i h_i(x)\}$

Subject to: $u_i \geq 0, i = 1, \dots, m$

Variables: $\boldsymbol{u} \in \mathbb{R}^m$, $\mathbf{v} \in \mathbb{R}^\ell$

- **Inequality** constraints $g_i(x) \leq 0$ corresponds to **nonnegative** Lagrange multipliers u_i.
- *Equality* constraints $h_i(x) = 0$ corresponds to *unrestricted* Lagrange multipliers v_i.

Duality theorem: (Informal statement)

Under certain convexity assumptions and suitable constraint qualifications, the primal and dual problems have equal optimal objective values.

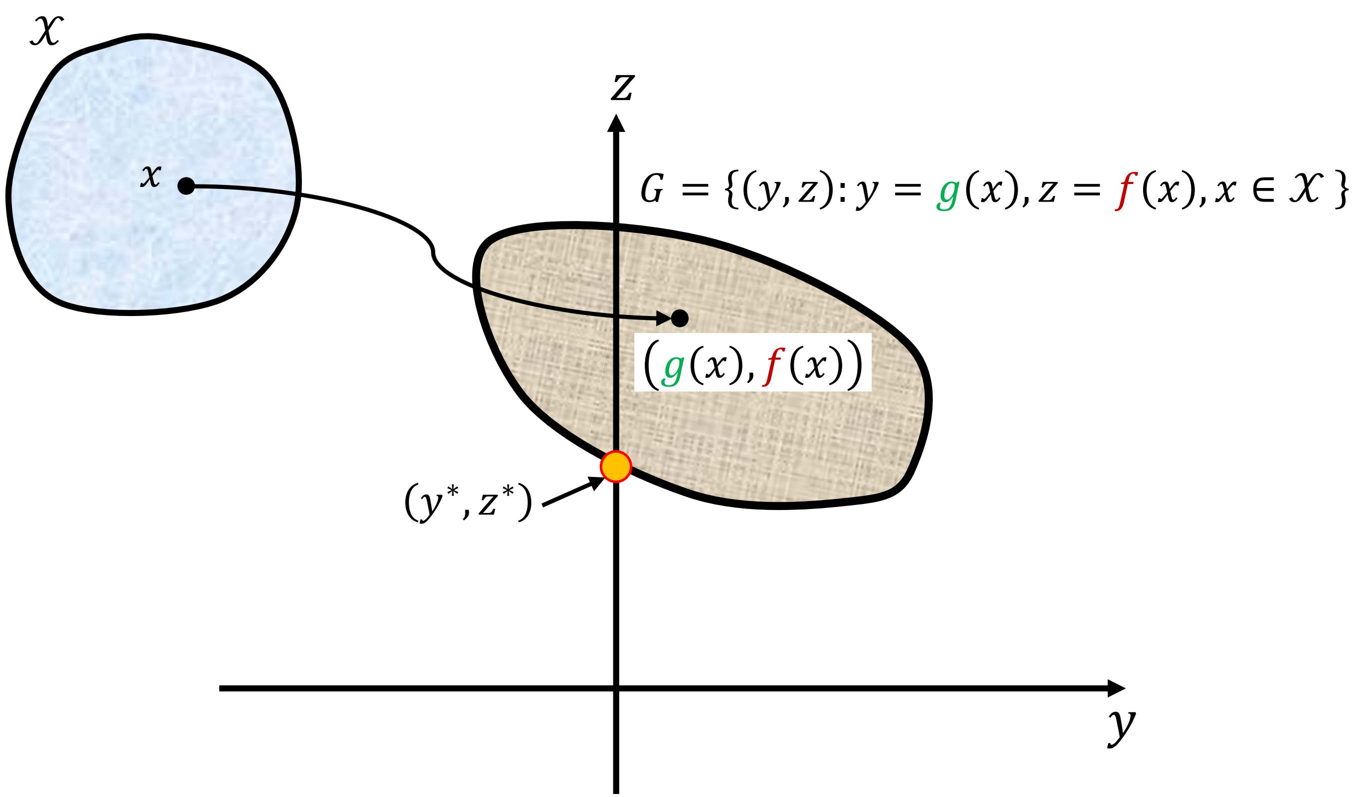
GEOMETRIC INTERPRETATION

Primal Problem P

Minimize $f(x)$

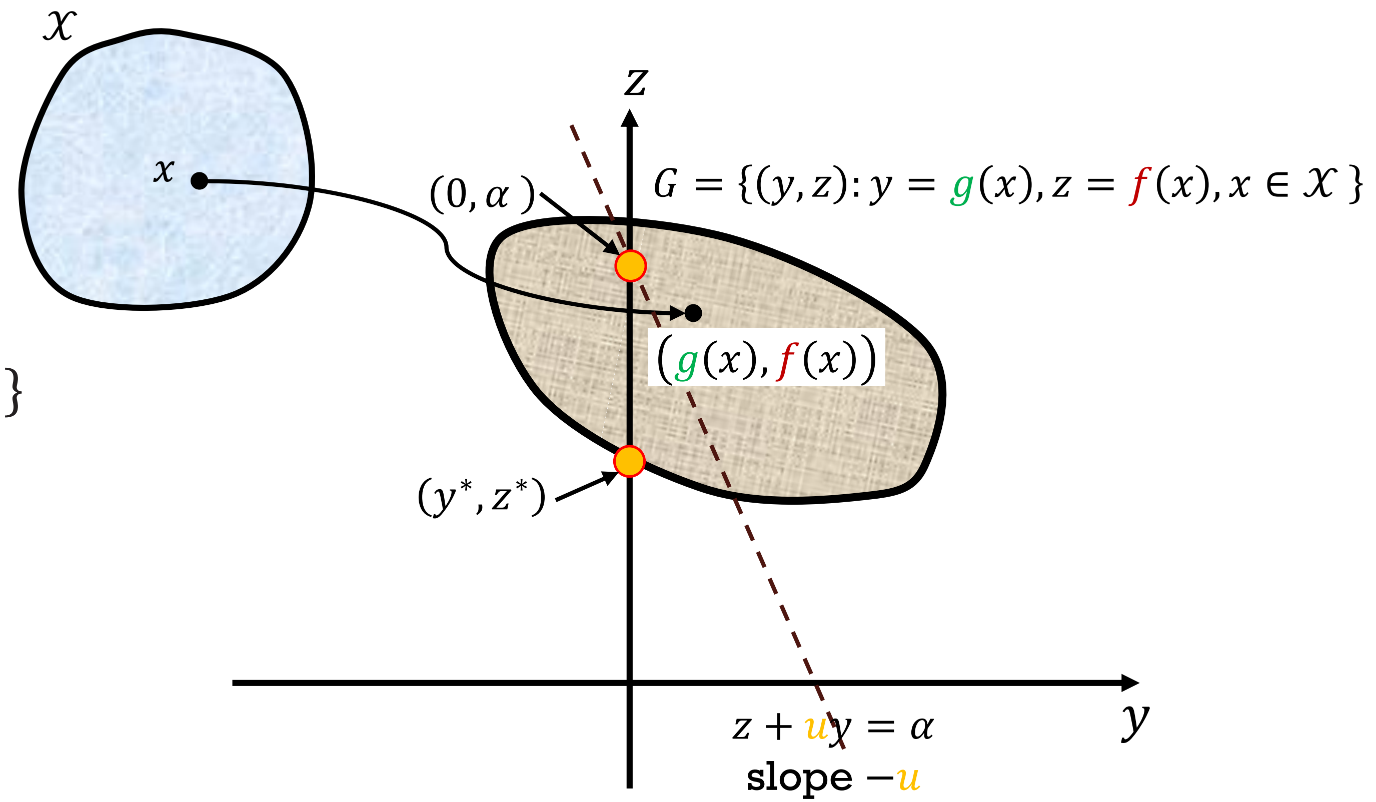
Subject to: $g(x) \le 0$

Variables: $x \in \mathcal{X}$



Primal minimum z^*

GEOMETRIC INTERPRETATION



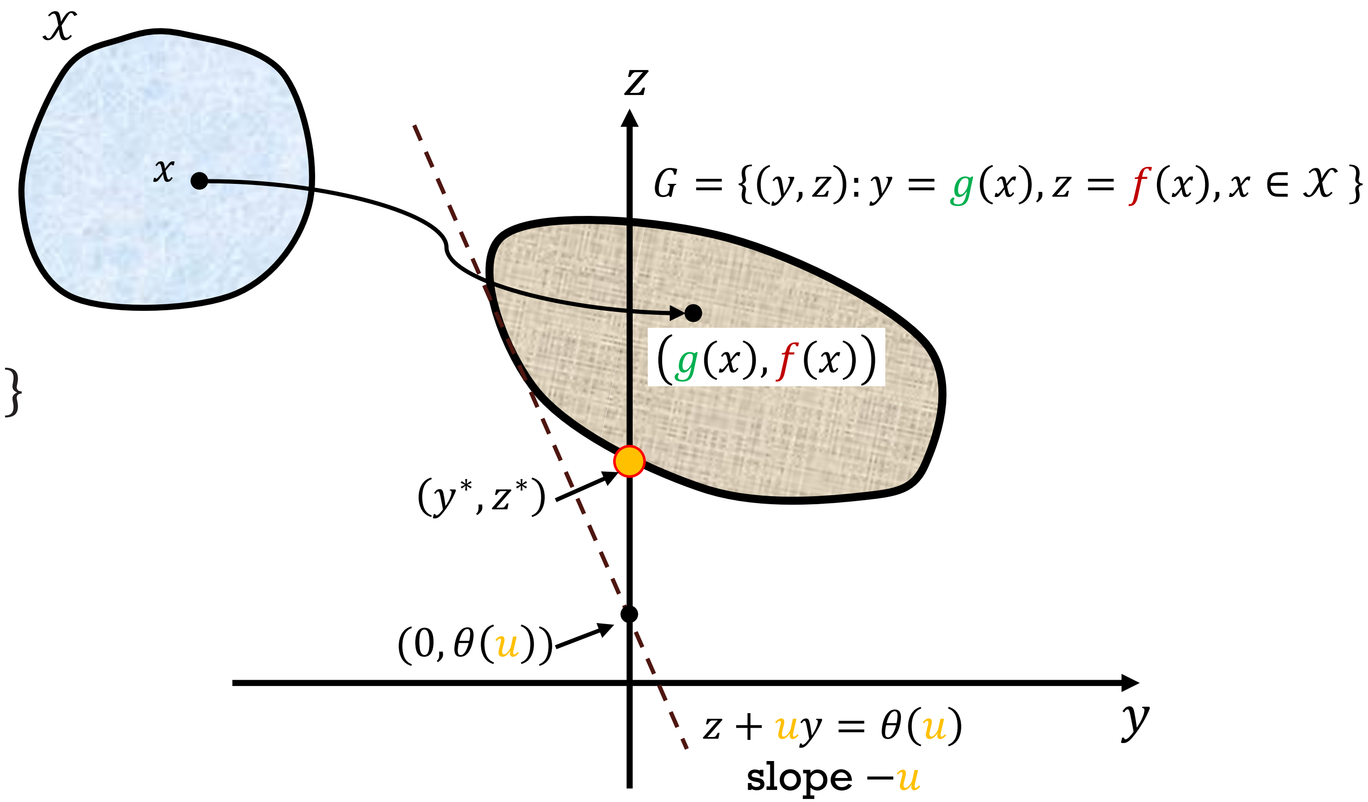
Lagrangian Dual Problem D

Maximize $\theta(u) = \inf_{x \in \mathcal{X}} \{f(x) + ug(x)\}$

Subject to: $u \geq 0$

Variables: $u \in \mathbb{R}$

GEOMETRIC INTERPRETATION



Lagrangian Dual Problem D

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GEOMETRIC INTERPRETATION

Lagrangian Dual Problem D

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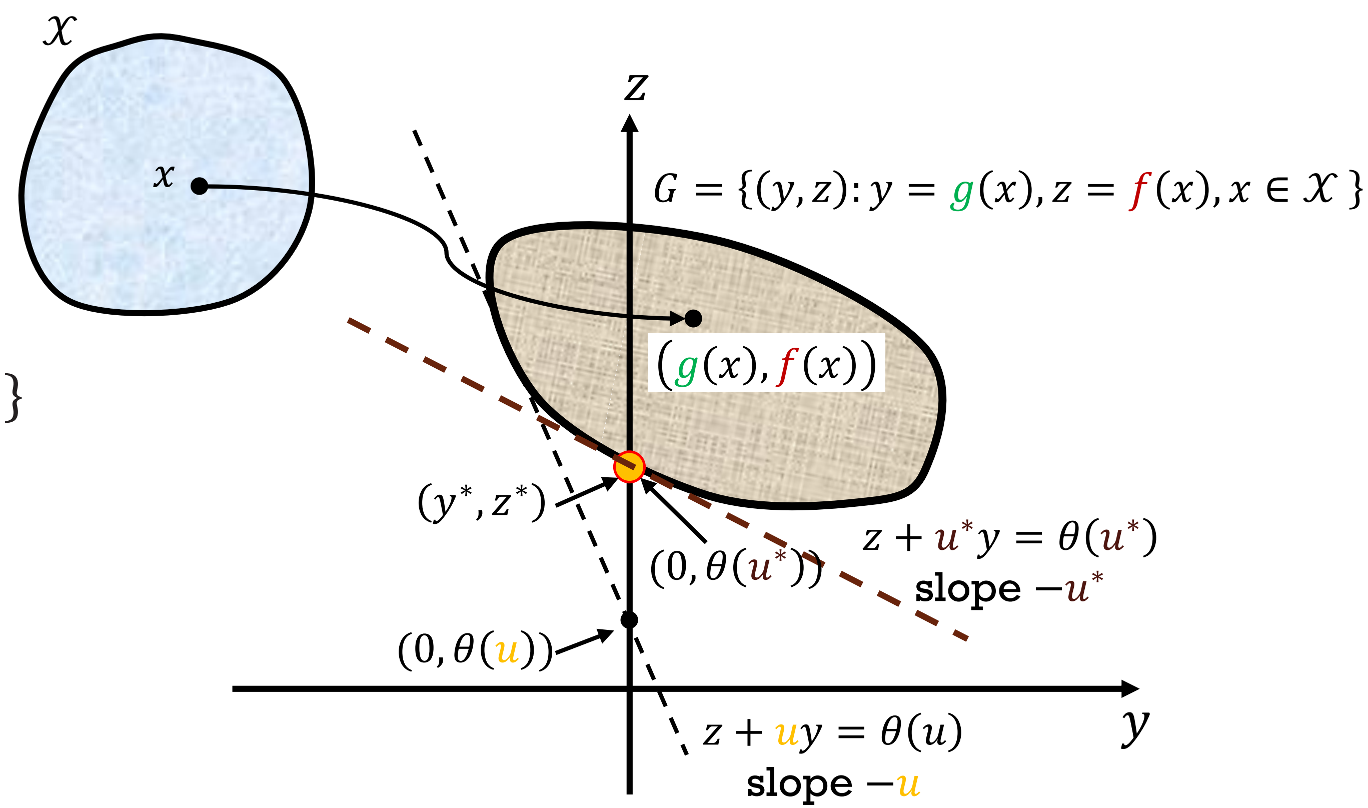


Primal minimum z^* = Dual maximum $\theta(u^*)$

WEAK DUALITY

Primal Problem P

Minimize $f(x)$

Subject to:

$g_i(x) \leq 0, i = 1, \dots, m$

$h_i(x) = 0, i = 1, \dots, \ell$

Variables: $x \in \mathcal{X}$

$\geq$

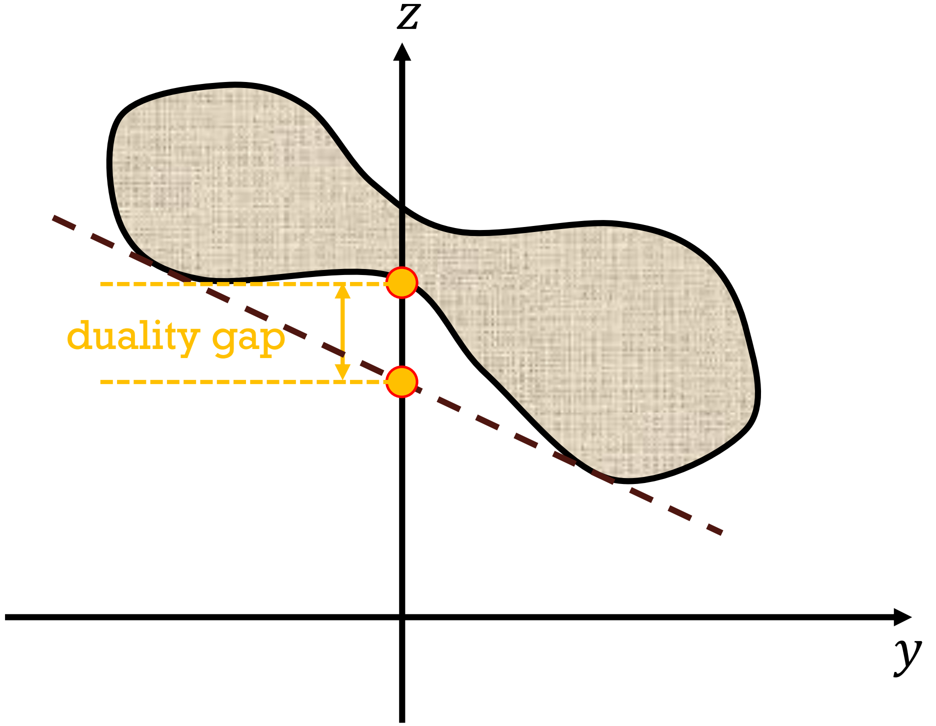
Lagrangian Dual Problem D

Maximize $\theta(\boldsymbol{u}, \mathbf{v}) = \inf_{x \in \mathcal{X}} \{f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{i=1}^{\ell} v_i h_i(x)\}$

Subject to: $u_i \geq 0, i = 1, \dots, m$

Variables: $\boldsymbol{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^\ell$

Duality gap: Difference between primal minimum and dual maximum



Weak Duality Theorem:

For arbitrary primal feasible solution x and dual feasible solution $(\boldsymbol{u}, \mathbf{v})$, one has

$$f(x) \geq \theta(\boldsymbol{u}, \mathbf{v})$$

Proof:

$$\theta(\boldsymbol{u}, \mathbf{v}) = \inf_{\tilde{x} \in \mathcal{X}} \left\{ f(\tilde{x}) + \sum_{i=1}^{m} u_i g_i(\tilde{x}) + \sum_{i=1}^{\ell} v_i h_i(\tilde{x}) \right\}$$

$$\leq f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{i=1}^{\ell} v_i h_i(x) \leq f(x)$$

STRONG DUALITY

Primal Problem P

Minimize $f(x)$

Subject to:

$g_i(x) \le 0, i = 1, \dots, m$

$h_i(x) = 0, i = 1, \dots, \ell$

Variables: $x \in \mathcal{X}$

Under some assumptions

$=$

Lagrangian Dual Problem D

Maximize $\theta(\mathbf{u}, \mathbf{v}) = \inf_{x \in \mathcal{X}} \{f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{i=1}^{\ell} v_i h_i(x)\}$

Subject to: $u_i \ge 0, i = 1, \dots, m$

Variables: $\mathbf{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^\ell$

Theorem (Strong Duality Theorem)

Let X be a nonempty convex set in $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be convex, and $h : \mathbb{R}^n \to \mathbb{R}^\ell$ be affine. Suppose that the following constraint qualification is satisfied. There exists an $\hat{x} \in X$ such that $g(\hat{x}) < 0$ and $h(\hat{x}) = 0$, and $0 \in \text{int}\, h(X)$, where $h(X) = \{h(x) : x \in X\}$. Then,

$$\inf\{f(x) : x \in X, g(x) \le 0, h(x) = 0\} = \sup\{\theta(u, v) : u \ge 0\}, \qquad (5)$$

where $\theta(u, v) = \inf\{f(x) + u^\top g(x) + v^\top h(x) : x \in X\}$. Furthermore, if the inf is finite, then $\sup\{\theta(u, v) : u \ge 0\}$ is achieved at $(\bar{u}, \bar{v})$ with $\bar{u} \ge 0$. If the inf is achieved at $\bar{x}$, then $\bar{u}^\top g(\bar{x}) = 0$.